Quantum integrability of bosonic Massive Thirring model in continuum

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Abstract

By using a variant of the quantum inverse scattering method, commutation relations between all elements of the quantum monodromy matrix of bosonic Massive Thirring (BMT) model are obtained. Using those relations, the quantum integrability of BMT model is established and the S-matrix of two-body scattering between the corresponding quasi particles has been obtained. It is observed that for some special values of the coupling constant, there exists an upper bound on the number of quasi-particles that can form a quantum-soliton state of BMT model. We also calculate the binding energy for a N-soliton state of quantum BMT model.

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1 Introduction

Quantum integrable field models in 1+1 dimensions are objects of interest due to their close connections with different areas of physics as well as mathematics [1-10]. These integrable theories have played an important role in understanding the basic nonperturbative aspects of physical theories relevant in the realistic 3+1 dimensional models. Through quantum inverse scattering method (QISM) one can establish the integrability property of these models and obtain the spectrum as well as different correlation functions of the corresponding models [4].

Massive Thirring model in 1+1 dimensions has been widely studied as a toy counterpart to low energy QCD, since it does not include many of the complications arising in 3+1 dimensions. The study of a nonlocal massless Thirring model is relevant, not only from a purely field theoretical point of view but also because of its connection with the physics of strongly correlated systems in one spatial dimension. This model describes an ensemble of non-relativistic particles coupled through a 2-body forward-scattering potential and displays Luttinger-liquid behaviour [11] that can play a role in real 1-dimensional semiconductors [12].

Massive Thirring model in 1+1 dimensions can be treated through QISM for both bosonic and fermionic field operators [6]. In this article, we shall focus our attention to bosonic Massive Thirring (BMT) model. The classical version of BMT model is described by the Hamiltonian

$$\begin{aligned} H = \int_{-\infty}^{\infty} dx \Big[-\frac{i}{2}\Big\{ (\phi_1^* \frac{\partial \phi_1}{\partial x} - \frac{\partial \phi_1^*}{\partial x}\phi_1) - (\phi_2^* \frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_2^*}{\partial x}\phi_2) \Big\} \\ - (\phi_1^*\phi_2 + \phi_2^*\phi_1) - 4\xi \phi_1^*\phi_2^*\phi_2\phi_1 \Big] \end{aligned} \tag{1.1}$$

with the equal time Poisson bracket (PB) relations

$$\begin{aligned} \{\phi_1(x), \phi_1(y)\} = \{\phi_1^*(x), \phi_1^*(y)\} = 0, \qquad \{\phi_1(x), \phi_1^*(y)\} = -i\delta(x-y), \\ \{\phi_2(x), \phi_2(y)\} = \{\phi_2^*(x), \phi_2^*(y)\} = 0, \qquad \{\phi_2(x), \phi_2^*(y)\} = -i\delta(x-y)\,. \end{aligned} \tag{1.2}$$

It is well known that this BMT model is intimately connected with the derivative nonlinear Schrödinger (DNLS) model. In fact, one can generate the Lax operator of BMT model

by 'fusing' two Lax operators of DNLS model with different spectral parameters [13]. The integrability of the classical DNLS model, possesing ultralocal PB structure, can be established from the fact that the corresponding monodromy matrix satisfies the classical Yang Baxter equation [14]. The quantised version of this DNLS model also preserves the integrability property. By applying QISM, the quantum integrability of DNLS model is established and the Bethe eigenstates for all conserved quantities have been constructed [14,15].

In an earlier work by Kulish and Sklyanin [6], the Lax operator and the corresponding R-matrix for the quantum BMT model has been given, though the detailed calculations are not being explicitly shown. Moreover, the quantum Yang-Baxter equation (QYBE) at the infinite interval limit and hence the corresponding commutation relation between the creation and annihilation operators have not been studied. However, it is evident that taking the infinite interval limit of the monodromy matrix and corresponding QYBE is necessary to get the spectrum for the quantum version of the Hamiltonian (1.1). In this context it may be mentioned that, by applying a variant of the QISM [3] which is directly applicable to field theoretical models, the quantum DNLS model has been shown to be integrable [15,16]. The infinite interval limit of the corresponding QYBE enabled us to obtain the spectrum of all the conserved quantities including the Hamiltonian and also the two-particle S-matrix. Therefore, it is interesting to explore the integrability property of the quantum BMT model by using the same variant of QISM that we applied for DNLS model. In this article our aim is to establish such integrability property of quantum BMT model and to obtain the spectrum of all conserved quantities including the Hamiltonian.

The arrangement of this article is as follows. In Section 2, we consider the classical BMT model and evaluate the PB relations among the various elements of the corresponding monodromy matrix at the infinite interval limit. Using these PB relations, the integrability of the classical BMT model can be established in the Liouville sense. In this section we also derive the expressions for the classical conserved quantities of BMT model. In Section 3, we construct the quantum monodromy matrix of BMT model on a finite interval and derive the corresponding QYBE. In Section 4, we consider the infinite interval limit of QYBE and obtain the commutation relations among the various elements

of the corresponding quantum monodromy matrix. Such commutation relations allow us to construct exact eigenstates for the quantum conserved quantities of BMT model by using the prescription of algebraic Bethe ansatz. In particular we are able to obtain the spectrum for the quantum version of the Hamiltonian (1.1). Furthermore we obtain the commutation relation between creation and annihilation operators of quasi-particles associated with BMT model and find out the S-matrix of two-body scattering among such quasi-particles. In this section we also calculate the binding energy for a N-soliton state of the quantum BMT model. Section 5 is the concluding section.

2 Integrability of the classical Massive Thirring model

The classical version of BMT model is described by the Lax operator [6]

$$U(x,\lambda) = i \begin{pmatrix} \xi\{\rho_1(x)-\rho_2(x)\} - \frac{1}{4}\{\lambda^2 - \frac{1}{\lambda^2}\} & \xi\{\lambda\phi_1^*(x) - \frac{1}{\lambda}\phi_2^*(x)\} \\ \lambda\phi_1(x) - \frac{1}{\lambda}\phi_2(x) & -\xi\{\rho_1(x)-\rho_2(x)\} + \frac{1}{4}\{\lambda^2 - \frac{1}{\lambda^2}\} \end{pmatrix} \tag{2.1}$$

where $\rho_1(x) = \phi_1^*(x)\phi_1(x)$, $\rho_2(x) = \phi_2^*(x)\phi_2(x)$, λ is the spectral parameter and ξ is the coupling constant of the theory. The bosonic fields $\phi_1(x)$, $\phi_2(x)$ satisfy the PB relations (1.2) and vanish at $|x| \rightarrow \infty$ limit. The monodromy matrix on finite and infinite intervals are defined as

$$T_{x_1}^{x_2}(\lambda) = \mathcal{P}\exp\int_{x_1}^{x_2} U(x,\lambda)dx \tag{2.2}$$

and

$$T(\lambda) = \lim_{\substack{x_2\rightarrow+\infty \\ x_1\rightarrow-\infty}} e(-x_2,\lambda)\left\{\mathcal{P}\exp\int_{x_1}^{x_2} U(x,\lambda)dx\right\} e(x_1,\lambda) \tag{2.3}$$

respectively, where $\mathcal{P}$ denotes the path ordering and $e(x,\lambda) = e^{-\frac{i}{4}\{\lambda^2 - \frac{1}{\lambda^2}\}\sigma_3 x}$.

First, we want to investigate the symmetry properties of the monodromy matrix (2.3). It is easy to check that, the Lax operator (2.1) satisfies the relations

$$U(x,\lambda)^* = KU(x,\lambda^*)K, \qquad U(x,-\lambda) = K'U(x,\lambda)K', \tag{2.4a, b}$$

where $K = \begin{pmatrix} 0 & \sqrt{-\xi} \\ 1/\sqrt{-\xi} & 0 \end{pmatrix}$ and $K' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By using these relations, we find that the symmetries of the monodromy matrix $T(\lambda)$ (2.3) are given by

$$T(\lambda)^* = KT(\lambda^*)K, \qquad T(-\lambda) = K'T(\lambda)K' . \tag{2.5a,b}$$

Due to the relation (2.5a), $T(\lambda)$ can be expressed in a form

$$T(\lambda) = \begin{pmatrix} a(\lambda) & -\xi b^*(\lambda) \\ b(\lambda) & a^*(\lambda) \end{pmatrix} , \tag{2.6}$$

where λ is taken as a real parameter. Moreover, by using the symmetry relation (2.5b), it is easy to see that $a(-\lambda) = a(\lambda)$ and $b(-\lambda) = -b(\lambda)$. Therefore, it is sufficient to derive the PB relations among the elements of $T(\lambda)$ only for $\lambda \geq 0$.

Next, our aim is to calculate the classical conserved quantities of BMT model by using the approach described in Ref 2. From (2.2), one obtains the differential equation followed by the monodromy matrix $T_{x_1}^{x_2}(\lambda)$ as

$$\frac{\partial}{\partial x_2} T_{x_1}^{x_2}(\lambda) = U(x_2, \lambda) T_{x_1}^{x_2}(\lambda) . \tag{2.7}$$

Now, let us decompose the monodromy matrix in the form

$$T_{x_1}^{x_2}(\lambda) = \Big(1 + W(x_2, \lambda) \Big) \exp Z(x_2, x_1, \lambda) \Big(1 + W(x_1, \lambda) \Big) , \tag{2.8}$$

where $Z(x_2, x_1, \lambda)$ is a diagonal matrix and $W(x, \lambda)$ is a nondiagonal one. The Lax operator of the classical BMT model can be expressed as $U(x, \lambda) = U_d(x, \lambda) + U_{nd}(x, \lambda)$, where $U_d(x, \lambda)$ is the diagonal part and $U_{nd}(x, \lambda)$ is the non-diagonal part of $U(x, \lambda)$. Using the above expression of the Lax operator $U(x, \lambda)$ (2.1), the differential equation (2.7) can be decomposed into

$$\frac{dZ}{dx} = U_d + U_{nd} W , \qquad \frac{dW}{dx} - 2U_d W - U_{nd} + W U_{nd} W = 0 . \tag{2.9a,b}$$

The structure of the Lax operator (2.1) ensures that $W(x_2, \lambda)$ and $Z(x_2, x_1, \lambda)$ can be written in the form

$$W(x_2, \lambda) = -\xi w^*(x_2, \lambda) \sigma_+ + w(x_2, \lambda) \sigma_- ,$$

$$Z(x_2, x_1, \lambda) = z(x_2, x_1, \lambda) \sigma_3 .$$

Substituting eqns (2.6) and (2.8) in the expression (2.3), and using $W(x,\lambda) \to 0$ at $|x| \to \infty$ limit, one obtains,

$$\ln a(\lambda) = \lim_{\substack{x_2\to+\infty \\ x_1\to-\infty}} \{z(x_2,x_1,\lambda) + \frac{i\lambda^2}{4}(x_2-x_1)\}.$$

Substituting the explicit form of $z(x_2,x_1,\lambda)$ (as obtained by integrating eqn.(2.9a)) to the above expression, we get the following form of $\ln a(\lambda)$:

$$\ln a(\lambda) = i\xi \int_{-\infty}^{+\infty} \{\phi_1^*\phi_1 - \phi_2^*\phi_2\}dx + i\xi\lambda \int_{-\infty}^{+\infty} \phi_1^* w dx - \frac{i\xi}{\lambda} \int_{-\infty}^{+\infty} \phi_2^* w dx\,. \tag{2.10}$$

Next, we expand $w(x,\lambda)$ in inverse powers of λ as

$$w(x,\lambda) = \sum_{j=0}^{\infty} \frac{w_j}{\lambda^{2j+1}}\,.$$

Using the differential eqn.(2.9b) followed by $W(x,\lambda)$, the expansion coefficient w_js can be obtained explicitly in a recursive way. The first few nonzero w_js are given by

$$w_0 = -2\phi_1\,; \quad w_2 = 4i\phi_{1x} + 8\xi\phi_1(\phi_2^*\phi_2) + 2\phi_2\,.$$

Substituting w_js in the expression of $\ln a(\lambda)$ (2.10), one gets

$$\ln a(\lambda) = \sum_{n=0}^{\infty} \frac{iC_n}{\lambda^{2n}},$$

where C_ns represent an infinite set of conserved quantities. The first two of them are explicitly given by

$$C_0 = -\xi \int_{-\infty}^{+\infty} \{\,\phi_1^*\phi_1 + \phi_2^*\phi_2\,\}dx\,, \tag{2.11a}$$

$$C_1 = 4i\xi \int_{-\infty}^{+\infty} \phi_1^*\phi_{1x}\,dx + 2\xi \int_{-\infty}^{+\infty} \{\phi_1^*\phi_2 + \phi_2^*\phi_1\}dx + 8\xi^2 \int_{-\infty}^{+\infty} (\phi_1^*\phi_1)(\phi_2^*\phi_2)dx\,. \tag{2.11b}$$

Next we expand $w(x,\lambda)$ in powers of λ as

$$w(x,\lambda) = \sum_{j=0}^{\infty} \tilde{w}_j\lambda^{2j+1}.$$

In a similar way as above, using (2.9b), the first few nonzero $\tilde{w}_j$s can be obtained as

$$\tilde{w}_0 = -2\phi_2 \qquad \tilde{w}_2 = -4i\phi_{2x} + 8\xi(\phi_1^*\phi_1)\phi_2 + 2\phi_1\,.$$

Correspondingly, eqn.(2.10) yields

$$\ln a(\lambda) = \sum_{n=0}^{\infty} i\tilde{C}_n \lambda^{2n},$$

where $\tilde{C}_n$s represent another infinite set of conserved quantities. The first two of them are explicitly given by

$$\tilde{C}_0 = \xi \int_{-\infty}^{+\infty} \{\, \phi_1^*\phi_1 + \phi_2^*\phi_2 \,\} dx\,, \tag{2.12a}$$

$$\tilde{C}_1 = 4i\xi \int_{-\infty}^{+\infty} \phi_2^*\phi_{2x}\, dx - 2\xi \int_{-\infty}^{+\infty} \{\phi_1^*\phi_2 + \phi_2^*\phi_1\} dx - 8\xi^2 \int_{-\infty}^{+\infty} (\phi_1^*\phi_1)(\phi_2^*\phi_2) dx\,. \tag{2.12b}$$

Now by combining these two sets of conserved quantities, the mass, momentum and the Hamiltonian of classical BMT model can be expressed in the following way:

$$\begin{aligned} N &= -\frac{1}{2\xi}(C_0 - \tilde{C}_0) = \int_{-\infty}^{+\infty} (\, \phi_1^*\phi_1 + \phi_2^*\phi_2 \,) dx\,, \\ P &= -\frac{1}{4\xi}(C_1 + \tilde{C}_1) = \int_{-\infty}^{+\infty} (\phi_1^*\phi_{1x} + \phi_2^*\phi_{2x})\, dx \\ H &= -\frac{1}{4\xi}(C_1 - \tilde{C}_1) = \int_{-\infty}^{\infty} [\, -i(\phi_1^*\phi_{1x} - \phi_2^*\phi_{2x}) - \{\phi_1^*\phi_2 + \phi_2^*\phi_1\} - 4\xi\phi_1^*\phi_2^*\phi_2\phi_1]\, dx. \end{aligned}$$

Next, we want to derive the PB relations among the elements of $T(\lambda)$ (2.6). We apply the equal time PB relations (1.2) between the basic field variables to evaluate the PB relations among the elements of the Lax operator (2.1) and find that

$$\{U(x,\lambda) \overset{\otimes}{,} U(y,\mu)\} = [r(\lambda,\mu), U(x,\lambda)\otimes 1\!\!1 + 1\!\!1 \otimes U(y,\mu)]\ \delta(x-y)\,, \tag{2.13}$$

where

$$r(\lambda,\mu) = -\xi\, \{\, t^c\sigma_3\otimes\sigma_3 + s^c(\sigma_+\otimes\sigma_- + \sigma_-\otimes\sigma_+)\,\} \tag{2.14}$$

with $t^c = \frac{\lambda^2+\mu^2}{2(\lambda^2-\mu^2)}$, $\quad s^c = \frac{2\lambda\mu}{\lambda^2-\mu^2}$. Now, by using the eqns.(2.13) and (2.3), one obtains

$$\{T(\lambda) \overset{\otimes}{,} T(\mu)\} = r_+(\lambda,\mu) T(\lambda)\otimes T(\mu) - T(\lambda)\otimes T(\mu) r_-(\lambda,\mu)\,, \tag{2.15}$$

where

$$r_\pm = -\xi \left(t^c\sigma_3\otimes\sigma_3 + s^c_\pm\sigma_+\otimes\sigma_- + s^c_\mp\sigma_-\otimes\sigma_+ \right)\,,$$

with $s^c_{\pm} = \pm 2i\pi\lambda^2\delta(\lambda^2 - \mu^2)$. By substituting the symmetric form of $T(\lambda)$ (2.6) to eqn.(2.15) and comparing the individual elements in both sides, we obtain

$$\{a(\lambda), a(\mu)\} = 0\,, \quad \{a(\lambda), a^\dagger(\mu)\} = 0\,, \quad \{b(\lambda), b(\mu)\} = 0\,, \tag{2.16a,b,c}$$

$$\{a(\lambda), b(\mu)\} = \xi \left(\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2}\right) a(\lambda)b(\mu) - 2i\pi\xi\lambda^2\,\delta(\lambda^2-\mu^2)\, b(\lambda)a(\mu)\,, \tag{2.16d}$$

$$\{a(\lambda), b^*(\mu)\} = -\xi \left(\frac{\lambda^2+\mu^2}{\lambda^2-\mu^2}\right) a(\lambda)b^*(\mu) + 2i\pi\xi\lambda^2\,\delta(\lambda^2-\mu^2)\, b^*(\lambda)a(\mu)\,, \tag{2.16e}$$

$$\{b(\lambda), b^*(\mu)\} = -4i\pi\lambda^2\,\delta(\lambda^2-\mu^2)\,|a(\lambda)|^2\,. \tag{2.16f}$$

From eqn.(2.16a) it follows that all expansion coefficients occuring in the expansions of $\ln a(\lambda)$ will have vanishing PB relations among themselves. Hence, the following expressions will hold true

$$\{C_m, C_n\} = \{\tilde{C}_m, \tilde{C}_n\} = \{C_m, \tilde{C}_n\} = 0,$$

for all values of m and n. Since the mass, momentum and the Hamiltonian of the classical BMT model has been expressed in terms of the expansion coefficients C_n and $\tilde{C}_n$s, all of them will have vanishing PB relations among themselves. Thus the integrability property of the classical BMT model, described by the Hamiltonian (1.1), is established in the Liouville sense.

3 Commutation relations for the quantum monodromy matrix on a finite interval

By using a version of QISM which is directly applicable to field models [3], in this section we shall show that the quantum monodromy matrix of BMT model on a finite interval satisfies QYBE. The basic field operators of the quantum BMT model satisfy the following equal time commutation relations:

$$\begin{aligned} [\phi_1(x), \phi_1(y)] &= \left[\phi_1^\dagger(x), \phi_1^\dagger(y)\right] = 0; \qquad \left[\phi_1(x), \phi_1^\dagger(y)\right] = \hbar\delta(x-y)\,, \\ [\phi_2(x), \phi_2(y)] &= \left[\phi_2^\dagger(x), \phi_2^\dagger(y)\right] = 0; \qquad \left[\phi_2(x), \phi_2^\dagger(y)\right] = \hbar\delta(x-y)\,, \end{aligned} \tag{3.1}$$

and the vacuum state is defined through the relations $\phi_1(x)|0\rangle = \phi_2(x)|0\rangle = 0$.

In analogy with the classical Lax operator (2.1), we assume that the quantum Lax operator of BMT model is given by

$$\mathcal{U}_q(x,\lambda) = i\begin{pmatrix} f_1\rho_1(x) - f_2\rho_2(x) - \frac{\lambda^2}{4} + \frac{1}{4\lambda^2} & \xi\lambda\phi_1^\dagger(x) - \frac{\xi}{\lambda}\phi_2^\dagger(x) \\ \lambda\phi_1(x) - \frac{1}{\lambda}\phi_2(x) & -g_1\rho_1(x) + g_2\rho_2(x) + \frac{\lambda^2}{4} - \frac{1}{4\lambda^2} \end{pmatrix} \tag{3.2}$$

where $\rho_1(x) = \phi_1^\dagger(x)\phi_1(x)$, $\rho_2(x) = \phi_2^\dagger(x)\phi_2(x)$ and f_1, f_2, g_1, g_2 are four parameters which will be determined later in this section through QYBE. Using the Lax operator (3.2), the quantum monodromy matrix on a finite interval is defined as

$$\mathcal{T}_{x_1}^{x_2}(\lambda) = \ :\mathcal{P}exp\int_{x_1}^{x_2}\mathcal{U}_q(x,\lambda)dx:, \tag{3.3}$$

where the symbol :: denotes the normal ordering of operators. This quantum monodromy matrix (3.3) satisfies a differential equation given by

$$\begin{aligned}\frac{\partial}{\partial x_2}\mathcal{T}_{x_1}^{x_2}(\lambda) \quad &=: \mathcal{U}_q(x_2,\lambda)\mathcal{T}_{x_1}^{x_2}(\lambda): \\ &= -\frac{i}{4}\{\lambda^2 - \frac{1}{\lambda^2}\}\sigma_3\mathcal{T}_{x_1}^{x_2}(\lambda) + i\xi\lambda\phi_1^\dagger(x_2)\sigma_+\mathcal{T}_{x_1}^{x_2}(\lambda) - \frac{i\xi}{\lambda}\phi_2^\dagger(x_2)\sigma_+\mathcal{T}_{x_1}^{x_2}(\lambda) \\ &\quad + i\lambda\sigma_-\mathcal{T}_{x_1}^{x_2}(\lambda)\phi_1(x_2) - \frac{i}{\lambda}\sigma_-\mathcal{T}_{x_1}^{x_2}(\lambda)\phi_2(x_2) \\ &\quad + if_1\phi_1^\dagger(x_2)e_{11}\mathcal{T}_{x_1}^{x_2}(\lambda)\phi_1(x_2) - if_2\phi_2^\dagger(x_2)e_{11}\mathcal{T}_{x_1}^{x_2}(\lambda)\phi_2(x_2) \\ &\quad - ig_1\phi_1^\dagger(x_2)e_{22}\mathcal{T}_{x_1}^{x_2}(\lambda)\phi_1(x_2) + ig_2\phi_2^\dagger(x_2)e_{22}\mathcal{T}_{x_1}^{x_2}(\lambda)\phi_2(x_2),\end{aligned} \tag{3.4}$$

where $e_{11} = \frac{1}{2}(1+\sigma_3)$ and $e_{22} = \frac{1}{2}(1-\sigma_3)$. Now, to apply QISM, we have to find out the differential equation satisfied by the product $\mathcal{T}_{x_1}^{x_2}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)$. By using the basic commutation relations (3.1) and the method of 'extension' [3], we find that the product of two monodromy matrices satisfies the following differential equation (detail calculations are given in Appendix A):

$$\frac{\partial}{\partial x_2}\left(\mathcal{T}_{x_1}^{x_2}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)\right) = \vdots\mathcal{L}(x_2;\lambda,\mu)\mathcal{T}_{x_1}^{x_2}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)\vdots\,, \tag{3.5}$$

where

$$\mathcal{L}(x;\lambda,\mu) = \mathcal{U}_q(x,\lambda)\otimes\mathbb{1} + \mathbb{1}\otimes\mathcal{U}_q(x,\mu) + \mathcal{L}_\triangle(x;\lambda,\mu)\,, \tag{3.6}$$

with

$$
\mathcal{L}_\triangle(x;\lambda,\mu) = \begin{pmatrix}
\begin{matrix} -\hbar f_1^2\rho_1(x) \\ -\hbar f_2^2\rho_2(x) \end{matrix} & \begin{matrix} -\hbar\xi\mu f_1\phi_1^\dagger(x) \\ -\frac{\hbar\xi}{\mu} f_2\phi_2^\dagger(x) \end{matrix} & 0 & 0 \\
 & & & \\
0 & \begin{matrix} \hbar g_1 f_1\rho_1(x) \\ +\hbar g_2 f_2\rho_2(x) \end{matrix} & 0 & 0 \\
 & & & \\
\begin{matrix} -\hbar\lambda f_1\phi_1(x) \\ -\frac{\hbar f_2}{\lambda}\phi_2(x) \end{matrix} & -\hbar\xi\{\lambda\mu + \frac{1}{\lambda\mu}\} & \begin{matrix} \hbar g_1 f_1\rho_1(x) \\ +\hbar g_2 f_2\rho_2(x) \end{matrix} & \begin{matrix} \hbar\xi\mu g_1\phi_1^\dagger(x) \\ +\frac{\hbar\xi g_2}{\mu}\phi_2^\dagger(x) \end{matrix} \\
 & & & \\
0 & \begin{matrix} \hbar\lambda g_1\phi_1(x) \\ +\frac{\hbar g_2}{\lambda}\phi_2(x) \end{matrix} & 0 & \begin{matrix} -\hbar g_1^2\rho_1(x) \\ -\hbar g_2^2\rho_2(x) \end{matrix}
\end{pmatrix} .
$$

In the expression (3.5), the sign of normal arrangement of operator factors is taken as $\vdots\vdots$. The sign $\vdots\vdots$, applied to the product of several operator factors (including $\phi_1, \phi_2, \phi_1^\dagger$ and $\phi_2^\dagger$), ensures the arrangement of all $\phi_1^\dagger$, $\phi_2^\dagger$ on the left, and all ϕ_1, ϕ_2 on the right, *without altering the order of the remaining factors*. For example,

$$
\vdots X\phi_1\phi_2\phi_1^\dagger\phi_2^\dagger Y \vdots = \phi_1^\dagger\phi_2^\dagger XY\phi_1\phi_2 \, ,
$$

where X and Y may in general be taken as some functions of the basic field operators.

Now one can easily check that $\mathcal{L}(x;\lambda,\mu)$ (3.6) follows an equation given by

$$
R(\lambda,\mu)\mathcal{L}(x;\lambda,\mu) = \mathcal{L}(x;\mu,\lambda)R(\lambda,\mu) \, , \tag{3.7}
$$

where $R(\lambda,\mu)$ is a (4×4) matrix of the form

$$
R(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s(\lambda,\mu) & t(\lambda,\mu) & 0 \\ 0 & t(\lambda,\mu) & s(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{3.8}
$$

with $t(\lambda,\mu) = \frac{\lambda^2-\mu^2}{\lambda^2 q - \mu^2 q^{-1}}$, $s(\lambda,\mu) = \frac{(q-q^{-1})\lambda\mu}{\lambda^2 q-\mu^2 q^{-1}}$ and $q = e^{-i\alpha}$. The above equation (3.7) enables us to determine the exact expressions of the parameters f_1, f_2, g_1, g_2, α in terms of the coupling constant ξ. We obtain :

$$
\hbar\xi = -\sin\alpha, \qquad f_1 = g_2 = \frac{\xi e^{-i\alpha/2}}{\cos\alpha/2}, \qquad g_1 = f_2 = \frac{\xi e^{i\alpha/2}}{\cos\alpha/2} . \tag{3.9a, b, c}
$$

Using eqns.(3.5) and (3.7), we find that the monodromy matrix (3.3) satisfies QYBE given by

$$
R(\lambda,\mu)\mathcal{T}_{x_1}^{x_2}(\lambda) \otimes \mathcal{T}_{x_1}^{x_2}(\mu) = \mathcal{T}_{x_1}^{x_2}(\mu) \otimes \mathcal{T}_{x_1}^{x_2}(\lambda) R(\lambda,\mu) \, . \tag{3.10}
$$

Using the above QYBE (3.10), the commutation relations among all elements of the quantum monodromy matrix (3.3) can be obtained easily.

Eqns. (3.9a,b,c), describing the relations between f_1, f_2, g_1, g_2, α and the coupling constant ξ, provide the necessary conditions for the Lax operator (3.2) to satisfy QYBE (3.10). From eqn.(3.9a) we can conclude that, the above method of deriving QYBE for quantum BMT model is applicable only when the coupling constant ξ lies within the range $|\xi| \leq \frac{1}{\hbar}$. The parameter α has a one-to-one correspondence with the coupling constant ξ for $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$. For the purpose of investigating the classical limit of the quantum Lax operator (3.2), we take the $\alpha \to 0$ limit which is equivalent to the $\hbar \to 0$ limit for a fixed value of ξ. From eqns.(3.9b,c), it follows that at this limit $f_1, f_2 \to \xi$ and $g_1, g_2 \to \xi$. Hence we find that the quantum Lax operator (3.2) correctly reproduces the classical Lax operator (2.1) at $\hbar \to 0$ limit.

4 Algebraic Bethe ansatz for the quantum monodromy matrix on an infinite interval

The quantum monodromy matrix in an infinite interval is defined as

$$\mathcal{T}(\lambda) = \lim_{\substack{x_2 \to +\infty \\ x_1 \to -\infty}} e(-x_2, \lambda) \mathcal{T}_{x_1}^{x_2}(\lambda) e(x_1, \lambda) \,, \tag{4.1}$$

where $\mathcal{T}_{x_1}^{x_2}(\lambda)$ is given by eqn.(3.3). Just as in the classical case, the quantum Lax operator (3.2) also satisfies the symmetry relations

$$\mathcal{U}_q(x,\lambda)^* = K\,\mathcal{U}_q(x,\lambda^*)\,K, \quad \mathcal{U}_q(x,-\lambda) = K'\,\mathcal{U}_q(x,\lambda)\,K' \,, \qquad (4.2a,b)$$

where K and K' matrices have appeared earlier in eqn.(2.4). Using eqn.(4.2a), the quantum monodromy matrix (4.1) can be expressed in a symmetric form given by

$$\mathcal{T}(\lambda) = \begin{pmatrix} A(\lambda) & -\xi B^\dagger(\lambda) \\ B(\lambda) & A^\dagger(\lambda) \end{pmatrix} , \tag{4.3}$$

where λ is a real parameter. From eqn.(4.2b), it follows that $A(-\lambda) = A(\lambda)$ and $B(-\lambda) = -B(\lambda)$. So it is sufficient to obtain the commutation relations among the elements of the quantum monodromy matrix (4.3) only for $\lambda \geq 0$.

Now we aim to obtain the infinite interval limit of the QYBE satisfied by $\mathcal{T}(\lambda)$ (4.3). To this end, we split the $\mathcal{L}(x;\lambda,\mu)$ matrix (3.6) into two parts:

$$\mathcal{L}(x;\lambda,\mu) = \mathcal{L}_0(\lambda,\mu) + \mathcal{L}_1(x;\lambda,\mu)\,,$$

where $\mathcal{L}_0(\lambda,\mu)$ is given by

$$\mathcal{L}_0(\lambda,\mu) = \lim_{|x|\to\infty} \mathcal{L}(x;\lambda,\mu) = \begin{pmatrix} \begin{matrix}-\frac{i}{4}(\lambda^2+\mu^2)\\ +\frac{i}{4}(\frac{1}{\lambda^2}+\frac{1}{\mu^2})\end{matrix} & 0 & 0 & 0 \\ 0 & \begin{matrix}-\frac{i}{4}(\lambda^2-\mu^2)\\ +\frac{i}{4}(\frac{1}{\lambda^2}-\frac{1}{\mu^2})\end{matrix} & 0 & 0 \\ 0 & -\hbar\xi\lambda\mu - \frac{\hbar\xi}{\lambda\mu} & \begin{matrix}\frac{i}{4}(\lambda^2-\mu^2)\\ -\frac{i}{4}(\frac{1}{\lambda^2}-\frac{1}{\mu^2})\end{matrix} & 0 \\ 0 & 0 & 0 & \begin{matrix}\frac{i}{4}(\lambda^2+\mu^2)\\ -\frac{i}{4}(\frac{1}{\lambda^2}+\frac{1}{\mu^2})\end{matrix} \end{pmatrix},$$

and $\mathcal{L}_1(x;\lambda,\mu)$ is the field dependent part of $\mathcal{L}(x;\lambda,\mu)$, which vanishes at $x\to\pm\infty$. From eqn.(3.7) we get

$$R(\lambda,\mu)\varepsilon(x;\lambda,\mu) = \varepsilon(x;\mu,\lambda)R(\lambda,\mu)\,, \tag{4.4}$$

where $\varepsilon(x;\lambda,\mu) = e^{\mathcal{L}_0(\lambda,\mu)x}$. By using the above mentioned splitting of $\mathcal{L}(x;\lambda,\mu)$, we derive the integral form of differential equation (3.5) as

$$\mathcal{T}_{x_1}^{x_2}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu) = \varepsilon(x_2-x_1;\lambda,\mu) + \int_{x_1}^{x_2} dx\,\varepsilon(x_2-x;\lambda,\mu)\,\vdots\mathcal{L}_1(x,\lambda,\mu)\mathcal{T}_{x_1}^{x}(\lambda)\otimes\mathcal{T}_{x_1}^{x}(\mu)\vdots\,.$$

From this integral relation it is clear that at the asymptotic limit $x_1,x_2\to\pm\infty$, $\mathcal{T}_{x_1}^{x_2}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)\to\varepsilon(x_2-x_1;\lambda,\mu)$, which is an oscillatory term. To get rid of this problem, we define an operator like

$$W(\lambda,\mu) = \lim_{\substack{x_2\to+\infty\\ x_1\to-\infty}} \varepsilon(-x_2;\lambda,\mu)\mathcal{T}_{x_1}^{x_2}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)\varepsilon(x_1;\lambda,\mu)\,. \tag{4.5}$$

In the above defined operator, the oscillatory nature of $\mathcal{T}_{x_1}^{x_2}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)$ has been removed and $W(\lambda,\mu)$ is perfectly well behaved at the limit $x_1,x_2\to\pm\infty$. By using (3.10) and (4.4), it is easy to verify that the operator $W(\lambda,\mu)$ (4.5) satisfies an equation given by

$$R(\lambda,\mu)W(\lambda,\mu) = W(\mu,\lambda)R(\lambda,\mu)\,. \tag{4.6}$$

The above equation represents the QYBE of BMT model at an infinite interval limit.

Next, we want to express the QYBE (4.6) directly in terms of the monodromy matrices (4.1) defined in an infinite interval. For this purpose, $W(\lambda,\mu)$ (4.5) can be rewritten as

$$W(\lambda,\mu) = C_+(\lambda,\mu)\mathcal{T}(\lambda)\otimes\mathcal{T}(\mu)C_-(\lambda,\mu)\,, \tag{4.7}$$

where

$$C_+(\lambda,\mu) = \lim_{x\to\infty}\varepsilon(-x;\lambda,\mu)E(x;\lambda,\mu), \quad C_-(\lambda,\mu) = \lim_{x\to-\infty}E(-x;\lambda,\mu)\varepsilon(x;\lambda,\mu)\,, \quad (4.8a,b)$$

with $E(x;\lambda,\mu) = e(x,\lambda)\otimes e(x,\mu)$. Substituting the explicit forms of $E(x;\lambda,\mu)$ and $\varepsilon(x;\lambda,\mu)$ to (4.8a,b), and taking the limits in the principal value sense: $\lim_{x\to\pm\infty}P(\frac{e^{ikx}}{k}) = \pm i\pi\delta(k)$, we obtain

$$C_+(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \rho_+(\lambda,\mu) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_-(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \rho_-(\lambda,\mu) & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.9}$$

where

$$\begin{aligned} \rho_\pm(\lambda,\mu) &= \mp\frac{2i\hbar\xi\left(\lambda\mu+\frac{1}{\lambda\mu}\right)}{\lambda^2-\mu^2-\frac{1}{\lambda^2}+\frac{1}{\mu^2}} + 2\pi\hbar\xi\left(\lambda\mu+\frac{1}{\lambda\mu}\right)\delta(\lambda^2-\mu^2-\frac{1}{\lambda^2}+\frac{1}{\mu^2}) \\ &= \mp\frac{2i\hbar\xi\{\lambda\mu+\frac{1}{\lambda\mu}\}}{\lambda^2-\mu^2-\frac{1}{\lambda^2}+\frac{1}{\mu^2}\mp i\epsilon}\,. \end{aligned} \tag{4.10}$$

Substituting the expression of $W(\lambda,\mu)$ (4.7) in eqn.(4.6), we can express this QYBE for the infinite interval in the form

$$R(\lambda,\mu)C_+(\lambda,\mu)\mathcal{T}(\lambda)\otimes\mathcal{T}(\mu)C_-(\lambda,\mu) = C_+(\mu,\lambda)\mathcal{T}(\mu)\otimes\mathcal{T}(\lambda)C_-(\mu,\lambda)R(\lambda,\mu)\,. \tag{4.11}$$

By inserting the explicit forms of $R(\lambda,\mu)$ (3.8), $C_\pm(\lambda,\mu)$ (4.9), and $\mathcal{T}(\lambda)$ (4.3) to the above QYBE (4.11) and comparing the matrix elements from both sides of this equation, we obtain the following commutation relations:

$$[A(\lambda),A(\mu)] = 0\,, \quad \left[A(\lambda),A^\dagger(\mu)\right] = 0\,, \quad [B(\lambda),B(\mu)] = 0\,, \quad (4.12a,b,c)$$

$$\begin{aligned} A(\lambda)B^\dagger(\mu) &= \frac{\mu^2q-\lambda^2q^{-1}}{\mu^2-\lambda^2-i\epsilon}B^\dagger(\mu)A(\lambda) \\ &= \frac{\mu^2q-\lambda^2q^{-1}}{\mu^2-\lambda^2}B^\dagger(\mu)A(\lambda) - 2\pi\hbar\xi\lambda\mu\delta(\lambda^2-\mu^2)B^\dagger(\lambda)A(\mu)\,, \end{aligned} \quad (4.12d)$$

$$B(\mu)A(\lambda) = \frac{\mu^2 q - \lambda^2 q^{-1}}{\mu^2 - \lambda^2 - i\epsilon} A(\lambda)B(\mu)$$

$$= \frac{\mu^2 q - \lambda^2 q^{-1}}{\mu^2 - \lambda^2} A(\lambda)B(\mu) - 2\pi\hbar\xi\lambda\mu\delta(\lambda^2 - \mu^2)A(\mu)B(\lambda)\,, \qquad (4.12e)$$

$$B(\mu)B^\dagger(\lambda) = \tau(\lambda,\mu)B^\dagger(\lambda)B(\mu) + 4\pi\hbar\xi\lambda\mu\delta(\lambda^2 - \mu^2)A^\dagger(\lambda)A(\lambda)\,, \qquad (4.12f)$$

where

$$\tau(\lambda,\mu) = \left[1 + \frac{8\hbar^2\xi^2\lambda^2\mu^2}{(\lambda^2-\mu^2)^2} - \frac{4\hbar^2\xi^2\{\lambda\mu + \frac{1}{\lambda\mu}\}^2}{(\lambda^2 - \mu^2 - \frac{1}{\lambda^2} + \frac{1}{\mu^2} - i\epsilon)(\lambda^2 - \mu^2 - \frac{1}{\lambda^2} + \frac{1}{\mu^2} + i\epsilon)}\right].$$

It is interesting to note that, for the case $\lambda \neq \mu$, eqn.(4.12f) gives $[B(\lambda), B^\dagger(\mu)] \neq 0$, whereas from eqn.(2.16f), one obtains that $\{b(\lambda), b^*(\mu)\} = 0$ for $\lambda \neq \mu$. Thus the correspondence principle is not manifest here in a straightforward manner. However the $\hbar \to 0$ limit of $\tau(\lambda,\mu)$, gives the correct classical counterpart of the commutation relation (4.12f).

Due to eqn.(4.12a), all the operator valued coefficients occuring in the expansion of $\ln A(\lambda)$ will commute among themselves. As a consequence the BMT model described by the Lax operator (3.2) turn out to be a quantum integrable system. By applying the method of algebraic Bethe ansatz, one can also construct the exact eigenstates for all commuting operators which are generated through the expansion of $\ln A(\lambda)$. With the help of eqn.(4.1), it is easy to find that $A(\lambda)|0\rangle = |0\rangle$. By using this relation and eqn.(4.12d), it can be shown that

$$A(\lambda)\,|\mu_1, \mu_2, \cdots, \mu_N\rangle = \prod_{r=1}^{N} \left(\frac{\mu_r^2 q - \lambda^2 q^{-1}}{\mu_r^2 - \lambda^2 - i\epsilon} \right) |\mu_1, \mu_2, \cdots, \mu_N\rangle\,, \qquad (4.13)$$

where μ_js are all distinct real or complex numbers and $|\mu_1, \mu_2, \cdots, \mu_N\rangle \equiv B^\dagger(\mu_1)B^\dagger(\mu_2)\cdots B^\dagger(\mu_N)|0\rangle$ represents a Bethe eigenstate. Using the commutation relation (4.12f) one can also calculate the norm of the eigenstates $B^\dagger(\mu_1)B^\dagger(\mu_2)\cdots B^\dagger(\mu_N)|0\rangle$. However, the commutation relation (4.12f) contains product of singular functions $(\lambda^2 - \mu^2 - \frac{1}{\lambda^2} + \frac{1}{\mu^2} - i\epsilon)^{-1}(\lambda^2 - \mu^2 - \frac{1}{\lambda^2} + \frac{1}{\mu^2} + i\epsilon)^{-1}$, which is undefined at the limit $\lambda \to \mu$. As a result, eigenstates like $B^\dagger(\mu_1)B^\dagger(\mu_2)\cdots B^\dagger(\mu_N)|0\rangle$ are not normalised on the δ-function. To solve this problem, we consider a reflection operator given by

$$R^\dagger(\lambda) \;=\; B^\dagger(\lambda)\left(A^\dagger(\lambda)\right)^{-1} \qquad (4.14)$$

and its adjoint $R(\lambda)$. By using eqns.(4.12a-f), we find that such reflection operators satisfy well defined commutation relations like

$$
\begin{aligned}
&R^\dagger(\lambda)R^\dagger(\mu) = S^{-1}(\lambda,\mu)\, R^\dagger(\mu)R^\dagger(\lambda)\,, \\
&R(\lambda)R(\mu) = S^{-1}(\lambda,\mu)\, R(\mu)R(\lambda)\,, \\
&R(\lambda)R^\dagger(\mu) = S(\lambda,\mu)\, R^\dagger(\mu)R(\lambda) + 4\pi\hbar\lambda^2\delta(\lambda^2-\mu^2)\,,
\end{aligned}
\tag{4.15}
$$

where

$$
S(\lambda,\mu) \;=\; \frac{\lambda^2 q - \mu^2 q^{-1}}{\lambda^2 q^{-1} - \mu^2 q}\,. \tag{4.16}
$$

The $S(\lambda,\mu)$ defined above represents the nontrivial S-matrix element of two-body scattering among the related quasi-particles. We find that this $S(\lambda,\mu)$ satisfies the following conditions:

$$
S^{-1}(\lambda,\mu) = S(\mu,\lambda) = S^*(\lambda,\mu)\,, \tag{4.17}
$$

and remains nonsingular at the limit $\lambda \to \mu$. Consequently, the action of the operators like $R^\dagger(\lambda)$ on the vacuum would produce well defined states which can be normalised on the δ-function.

The point to be noted here is that in eqn.(4.13), the eigenvalues of $A(\lambda)$ are in general complex. To get real eigenvalues, we define a new operator $\ln \hat{A}(\lambda)$ through the relation $\ln \hat{A}(\lambda) \equiv \ln A(\lambda e^{\frac{-i\alpha}{2}})$ and expand this operator in inverse powers of λ:

$$
\ln \hat{A}(\lambda) = \sum_{n=0}^{\infty} \frac{i\mathcal{C}_n}{\lambda^{2n}}\,. \tag{4.18}
$$

Using eqns.(4.13) and (4.18), it is easy to see that $\mathcal{C}_n$s satisfy eigenvalue equations like

$$
\mathcal{C}_n \left|\mu_1,\mu_2,\cdots,\mu_N\right\rangle , = \chi_n \left|\mu_1,\mu_2,\cdots,\mu_N\right\rangle ,
$$

where the first few χ_ns are explicitly given by

$$
\chi_0 = \alpha N\,, \qquad \chi_1 = 2\sin\alpha \sum_{j=1}^{N}\mu_j^2\,, \quad \chi_2 = \sin 2\alpha \sum_{j=1}^{N}\mu_j^4\,. \tag{4.19}
$$

It may be noted that these eigenvalues are all real when μ_js are taken as real numbers. Next we expand the operator $\ln \hat{A}(\lambda)$ in powers of λ as

$$
\ln \hat{A}(\lambda) = \sum_{n=0}^{\infty} i\tilde{\mathcal{C}}_n \lambda^{2n}\,, \tag{4.20}
$$

and by using (4.13) we obtain

$$\tilde{\mathcal{C}}_n \left|\mu_1, \mu_2, \cdots, \mu_N\right\rangle , = \tilde{\chi}_n \left|\mu_1, \mu_2, \cdots, \mu_N\right\rangle .$$

The first few $\tilde{\chi}_n$s are explicitly given by

$$\tilde{\chi}_0 = -\alpha N \, , \qquad \tilde{\chi}_1 = -2\sin\alpha \sum_{j=1}^{N} \frac{1}{\mu_j^2} \, , \qquad \tilde{\chi}_2 = -\sin 2\alpha \sum_{j=1}^{N} \frac{1}{\mu_j^4} \, . \tag{4.21}$$

In analogy with the classical case, one can now define the momentum and Hamiltonian of the quantum BMT model as

$$\mathcal{P} = -\frac{1}{4\xi}(C_1 + \tilde{C}_1), \qquad \mathcal{H} = -\frac{1}{4\xi}\left(C_1 - \tilde{C}_1\right).$$

By using (4.19) and (4.21), the eigenvalue equations corresponding to the above momentum and Hamiltonian are obtained as

$$\begin{aligned} \mathcal{P}|\mu_1, \mu_2, \cdots, \mu_N\rangle &= \frac{1}{2}\sum_{j=1}^{N}\left(\mu_j^2 - \frac{1}{\mu_j^2}\right)\left|\mu_1, \mu_2, \cdots, \mu_N\right\rangle , \\ \mathcal{H}|\mu_1, \mu_2, \cdots, \mu_N\rangle &= \frac{1}{2}\sum_{j=1}^{N}\left(\mu_j^2 + \frac{1}{\mu_j^2}\right)\left|\mu_1, \mu_2, \cdots, \mu_N\right\rangle . \end{aligned} \tag{4.22}$$

In the above expressions, μ_js are taken as real numbers and $|\mu_1, \mu_2, \cdots, \mu_N\rangle$ represents a scattering state. Now to construct quantum N-soliton states of BMT model, complex values of μ_j can be chosen in such a way so that the eigenvalues corresponding to different expansion coefficients of $\ln \hat{A}(\lambda)$ still remains real. Such a choice is given by

$$\mu_j = \mu \exp\Big[-i\alpha \left(\frac{N+1}{2} - j\right)\Big], \tag{4.23}$$

where μ is a real parameter and $j \in [1, 2, \cdots, N]$. For the above choice of μ_j, eqn.(4.13) takes the form

$$A(\lambda)|\mu_1, \mu_2, \cdots, \mu_N\rangle = q^{-N}\Big(\frac{\lambda^2 - \mu^2 q^{N+1}}{\lambda^2 - \mu^2 q^{-N+1}}\Big)|\mu_1, \mu_2, \cdots, \mu_N\rangle . \tag{4.24}$$

Consequently, the energy eigenvalue equation corresponding to the quantum N-soliton state can be obtained as $\mathcal{H}|\mu_1, \mu_2, \cdots, \mu_N\rangle = E|\mu_1, \mu_2, \cdots, \mu_N\rangle$, where

$$E = \frac{1}{2}\left(\mu^2 + \frac{1}{\mu^2}\right)\frac{\sin\alpha N}{\sin\alpha}. \tag{4.25}$$

Thus we find that quantum N-soliton states can be constructed for BMT model for $N > 1$. Now we assume a particular value of the coupling constant ξ given by $\xi = -\sin\alpha = -\sin(\frac{2\pi m}{n})$, where m and n are nonzero integers which do not have any common factor. From eqn.(4.23), we obtain $\mu_j = \mu_{j+n}$ for the above choice of ξ. Since all the μ_js have to be distinct, we get $N \leq n$ as a restriction on the number of quasi-particles that can form a quantum soliton state for BMT model when $\xi = -\sin(\frac{2\pi m}{n})$.

Next we aim to calculate the binding energy for a N-soliton state of quantum BMT model. Substituting the expression of μ_j (4.23) to the first relation in eqn.(4.22), the momentum eigenvalue of a N-soliton state is obtained as

$$P = \frac{1}{2}(\mu^2 - \frac{1}{\mu^2})\frac{\sin\alpha N}{\sin\alpha}\,. \tag{4.26}$$

It is intersting to observe that the energy (4.25) and the momentum eigenvalue (4.26) of a N-soliton state satisfy the dispersion relation $E^2 = P^2 + m^2$, where $m = \frac{\sin\alpha N}{\sin\alpha}$. To calculate binding energy we assume that the momentum P (4.26) is equally distributed among N number of single-particle scattering states. The real wave number associated with each of these single particle states is denoted by μ_0. With the help of eqns.(4.22) and (4.26), we find that

$$\mu_0^2 - \frac{1}{\mu_0^2} = (\mu^2 - \frac{1}{\mu^2})\frac{\sin\alpha N}{N\sin\alpha}\,. \tag{4.27}$$

Using eqn.(4.22), the total energy for N number of such single particle states is obtained as

$$E' = \frac{N}{2}\left(\mu_0^2 + \frac{1}{\mu_0^2}\right) = \frac{N}{2}\left\{\left(\mu^2 - \frac{1}{\mu^2}\right)^2\frac{\sin^2\alpha N}{N^2\sin^2\alpha} + 4\right\}^{\frac{1}{2}}\,. \tag{4.28}$$

Subtracting E (4.25) from E' (4.28), we obtain the binding energy of the quantum N-soliton state as

$$\begin{aligned} E_B(\alpha, N) &= E' - E \\ &= \frac{N}{2}\left\{\left(\mu^2 - \frac{1}{\mu^2}\right)^2\frac{\sin^2\alpha N}{N^2\sin^2\alpha} + 4\right\}^{\frac{1}{2}} - \frac{1}{2}\left(\mu^2 + \frac{1}{\mu^2}\right)\frac{\sin\alpha N}{\sin\alpha}\,. \end{aligned} \tag{4.29}$$

Note that the above expression of $E_B(\alpha, N)$ remains invariant under the transformation $\alpha \to -\alpha$. So it is sufficient to analyse the nature of binding energy within the range

$0 < \alpha \leq \frac{\pi}{2}$. Now, for $E_B(\alpha, N)$ to represent the energy of a real bound state, E' has to be greater than E. Since E' (4.28) is always positive, it is evident that $E' > E$ for $E < 0$. So we will restrict our attention only for the case $E > 0$, when the condition $E' > E$ is equivalent to ${E'}^2 > E^2$. Substituting the explicit expressions for E' (4.28) and E (4.25), the above condition takes the form

$$N \sin \alpha > \sin \alpha N \,. \tag{4.30}$$

Substituting $N = 2$ in (4.30), we get the trivial inequality $1 > \cos \alpha$ for $\alpha > 0$. So the condition (4.30) is satisfied for $N = 2$ case within our chosen range of α. By using the method of induction, we can easily prove that the condition (4.30) is valid for arbitrary values of N. Thus we get an N-soliton bound state when α lies in the range $0 < |\alpha| \leq \frac{\pi}{2}$.

5 Concluding Remarks

In this article we consider the classical Lax opeartor of BMT model and obtain the PB relations among various elements of the classical monodromy matrix at the infinite interval limit. By using these PB relations, the classical integrability of BMT model is established in the Liouville sense. We also calculate the classical conserved quantities of BMT model. Next, we quantise the Lax opeartor of BMT model. By using a variant of QISM, that can be directly applied to the field theoretic models, we obtain the QYBE for the quantum monodromy matrix at a finite interval. This QYBE enables us to determine the various parameters of the quantum Lax operator in terms of the coupling constant ξ. Then we take the infinite interval limit of this QYBE and derive all possible commutation relations among the various elements of the corresponding quantum monodromy matrix. These commutation relations enable us to establish the quantum integrability of BMT model and also to construct the exact eigenstates for the quantum version of the Hamiltonian (1.1) as well as other conserved quantities by using algebraic Bethe ansatz. We also obtain the commutation relation between creation and annihilation operators associated with quasi-particles of BMT model and find out the S-matrix for two body scattering.

In this context, we consider the BMT model with some special values of coupling constant given by $\xi = -\sin \alpha = -\sin(\frac{2\pi m}{n})$, where m and n are nonzero integers with no

common factor. It turns out that the number of quasi-particles, which form a bound state for such quantum BMT model, cannot exceed the value of n. We have also derived the exact expression of binding energy for a N-soliton state of quantum BMT model. The binding energy turns out to be positive for all allowed values of α.

The commutation relation between creation and annihilation operators will play an important role in the future study, since by using it one might be able to calculate the norm of Bethe eigenstates and various correlation functions of the BMT model. In future, we would also like to obtain the quantum conserved quantities of BMT model in terms of the field operators by using a method which was used earlier in the case of nonlinear Schrodinger model [17] and DNLS model [16].

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Appendix A

Here we give the details of deriving eqn.(3.5). Direct attempt to calculate $\frac{\partial}{\partial x_2}\left(\mathcal{T}_{x_1}^{x_2}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)\right)$ by using eqn.(3.4), leads to indeterminate expressions of the form $\left[\mathcal{T}_{x_1}^{x_2}(\lambda),\phi_1^\dagger(x_2)\right]$ and $\left[\mathcal{T}_{x_1}^{x_2}(\lambda),\phi_2^\dagger(x_2)\right]$. To avoid this problem by using the method of extension [3], we shift the upper limit of the monodromy matrix $\mathcal{T}_{x_1}^{x_2}(\lambda)$ by a small amount ϵ and take $\epsilon\to 0$ limit only after differentiating the product $\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)$ with respect to x_2. So, using eqn.(3.4), we obtain

$$\begin{aligned}\frac{\partial}{\partial x_2}\left(\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)\right) &= \vdots\left(\mathcal{U}_q(x_2+\epsilon;\lambda)\otimes 1\!\!1 + 1\!\!1\otimes\mathcal{U}_q(x_2;\mu)\right)\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\mathcal{T}_{x_1}^{x_2}(\mu)\vdots \\ &\quad + K_+ + K_- \,, \qquad (A1)\end{aligned}$$

where

$$\begin{aligned}
K_+ &= i\xi\mu\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_1^\dagger(x_2)\right]\otimes\sigma_+\mathcal{T}_{x_1}^{x_2}(\mu) - \frac{i\xi}{\mu}\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_2^\dagger(x_2)\right]\otimes\sigma_+\mathcal{T}_{x_1}^{x_2}(\mu)\\
&+ if_1\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_1^\dagger(x_2)\right]\otimes e_{11}\mathcal{T}_{x_1}^{x_2}(\mu)\phi_1(x_2)\\
&- if_2\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_2^\dagger(x_2)\right]\otimes e_{11}\mathcal{T}_{x_1}^{x_2}(\mu)\phi_2(x_2)\\
&- ig_1\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_1^\dagger(x_2)\right]\otimes e_{22}\mathcal{T}_{x_1}^{x_2}(\mu)\phi_1(x_2)\\
&+ ig_2\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_2^\dagger(x_2)\right]\otimes e_{22}\mathcal{T}_{x_1}^{x_2}(\mu)\phi_2(x_2)\,,\\
K_- &= i\lambda\sigma_-\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\left[\phi_1(x_2+\epsilon),\mathcal{T}_{x_1}^{x_2}(\mu)\right] - \frac{i}{\lambda}\sigma_-\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\left[\phi_2(x_2+\epsilon),\mathcal{T}_{x_1}^{x_2}(\mu)\right]\\
&+ if_1\phi_1^\dagger(x_2+\epsilon)e_{11}\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\left[\phi_1(x_2+\epsilon),\mathcal{T}_{x_1}^{x_2}(\mu)\right]\\
&- if_2\phi_2^\dagger(x_2+\epsilon)e_{11}\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\left[\phi_2(x_2+\epsilon),\mathcal{T}_{x_1}^{x_2}(\mu)\right]\\
&- ig_1\phi_1^\dagger(x_2+\epsilon)e_{22}\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\left[\phi_1(x_2+\epsilon),\mathcal{T}_{x_1}^{x_2}(\mu)\right]\\
&+ ig_2\phi_2^\dagger(x_2+\epsilon)e_{22}\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda)\otimes\left[\phi_2(x_2+\epsilon),\mathcal{T}_{x_1}^{x_2}(\mu)\right]\,.
\end{aligned}$$

Now we consider the case, $\epsilon>0$. Since $\phi_1(x_2+\epsilon)$ and $\phi_2(x_2+\epsilon)$ commute with $\phi_1(x)$, $\phi_1^\dagger(x)$, $\phi_2(x)$, $\phi_2^\dagger(x)$ for all x lying within x_1 and x_2, we get $\left[\phi_1(x_2+\epsilon),\mathcal{T}_{x_1}^{x_2}(\mu)\right]=\left[\phi_2(x_2+\epsilon),\mathcal{T}_{x_1}^{x_2}(\mu)\right]=0$. Thus we can conclude that for a positive ϵ, $K_-=0$. So we have to calculate only the nontrivial commutators $\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_1^\dagger(x_2)\right]$ and $\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_2^\dagger(x_2)\right]$ appearing in the expression of K_+.

First let us calculate the commutator $\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda),\phi_1^\dagger(x_2)\right]$. For this purpose, we consider a ‘transformation’ Ω, which replaces the classical variables $\psi(x)$ and $\psi^*(x)$ by quantum

operators $\psi(x)$ and $\psi^\dagger(x)$ respectively. Next we use a correspondence principle [3],

$$\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda), \phi_1^\dagger(x_2)\right] = i\hbar : \Omega \left\{ T_{x_1}^{x_2+\epsilon}(q;\lambda), \phi_1^*(x_2)\right\} : , \qquad (A2)$$

where $T_{x_1}^{x_2+\epsilon}(q;\lambda)$ represents a classical monodromy matrix given by

$$T_{x_1}^{x_2+\epsilon}(q;\lambda) = \mathcal{P} \exp \int_{x_1}^{x_2} U_q(x,\lambda) dx \, ,$$

and $U_q(x,\lambda) = \Omega^{-1}\mathcal{U}_q(x,\lambda)$. By using the fundamental PB relations (1.2), it is easy to find that

$$\begin{aligned} \{T_{x_1}^{x_2+\epsilon}(q;\lambda), \phi_1^*(x_2)\} &= \int_{x_1}^{x_2+\epsilon} dx \; T_x^{x_2+\epsilon}(q;\lambda)\{U_q(x,\lambda), \phi_1^*(x_2)\} T_{x_1}^{x}(q,\lambda) \\ &= T_{x_2}^{x_2+\epsilon}(q;\lambda)(f_1\phi_1^*(x_2)e_{11} - g_1\phi_1^*(x_2)e_{22} + \lambda\sigma_-)T_{x_1}^{x_2}(q;\lambda) \, . \end{aligned}$$

Taking $\epsilon \to 0$ limit of the above expression and substituting it in (A.2), we obtain

$$\lim_{\epsilon\to 0} \left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda), \phi_1^\dagger(x_2) \right] = i\hbar \left(f_1\phi_1^\dagger(x_2)e_{11} - g_1\phi_1^\dagger(x_2)e_{22} + \lambda\sigma_- \right) \mathcal{T}_{x_1}^{x_2}(\lambda) \, . \qquad (A3)$$

Next we have to calculate the commutator $\left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda), \phi_2^\dagger(x_2) \right]$. Using the same correspondence principle as before and finally taking the $\epsilon \to 0$ limit one obtains,

$$\lim_{\epsilon\to 0} \left[\mathcal{T}_{x_1}^{x_2+\epsilon}(\lambda), \phi_2^\dagger(x_2) \right] = i\hbar \left(-f_2\phi_2^\dagger(x_2)e_{11} + g_2\phi_2^\dagger(x_2)e_{22} - \frac{1}{\lambda}\sigma_- \right) \mathcal{T}_{x_1}^{x_2}(\lambda) \, . \qquad (A4)$$

Taking the $\epsilon \to 0$ limit of eqn.(A1) and using (A3) and (A4), we finally obtain the differential equation (3.5). Note that, instead of $\epsilon > 0$, we could have chosen $\epsilon < 0$ in eqn.(A1). In that case only the commutators $\left[\phi_1(x_2+\epsilon), \mathcal{T}_{x_1}^{x_2}(\mu)\right]$ and $\left[\phi_2(x_2+\epsilon), \mathcal{T}_{x_1}^{x_2}(\mu)\right]$ give nontrivial contributions. However,by repeating similar steps as outlined above and finally taking the $\epsilon \to 0$ limit, we would have obtained the same differential equation (3.5).

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